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q-TRIGONAL KLEIN SURFACES

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ABSTRACT

In this paper q-trigonal Klein surfaces are introduced in a similar way to that of q-hyperelliptic surfaces. They are characterized by means of non-Euclidean crystallographic groups (NEC groups in short). As a consequence of this characterization, given a family of Klein surfaces (orientable or not) with topological genus g and k boundary components the admissible values for q are calculated. In particular, the families for which there is no admissible q or families with unique q are obtained.

1. Introduction

A Klein surface X is a surface equipped with a dianalytic structure. The modern study of Klein surfaces started with [1]. There is a Uniformization Theorem similar to that of Poincaré and Kobe for Riemann surfaces. A Klein surface X is the quotient $\mathcal{D}/\Gamma$, where $\mathcal{D}$ is the hyperbolic plane and Γ is a surface NEC group.

In the last three decades the study of the automorphism groups of Klein surfaces has been an important research field. A reference book about Klein surfaces and NEC groups is [6] with a long list of references. Particular families of Klein surfaces have been studied very much, for example hyperelliptic surfaces.

A Klein surface X is said to be q-**hyperelliptic** if and only if it admits an automorphism of order two, such that the quotient $X/<\phi>$ has algebraic genus

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q. When $q = 0$ the surface is said to be hyperelliptic and its characterization, by means of NEC groups, is given in [4]. The case $q = 1$ corresponds to **elliptic-hyperelliptic** surfaces [5]. The general case, q, is studied in [3] for planar surfaces and in [7] for orientable surfaces of genus 1.

A Klein surface X is said to be **cyclic trigonal** if and only if X admits an automorphism ϕ of order three such that $X/<\phi>$ has algebraic genus 0. Cyclic trigonal Klein surfaces and their automorphism groups have been studied in [2].

In this work we introduce q-trigonal Klein surfaces. Such a surface X admits an automorphism ϕ of order three, such that $X/<\phi>$ has algebraic genus q. Let us denote by $\mathcal{K}_{g,k}^{\pm}$ the family of Klein surfaces of topological genus g, k boundary components, orientable $(+)$ or not $(-)$. For each family we characterize in Section 3 the q-trigonality by means of NEC groups and we calculate the admissible values of q. As a consequence of this characterization we answer the following questions: what families do not contain any q-trigonal surface and which ones admit a unique admissible value q?

In the next Section we give necessary preliminaries about NEC groups and Klein surfaces.

2. Preliminaries

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{D}$ (including reversing-orientation isometries) with compact quotient $\mathcal{D}/\Gamma$ [13]. The signature of Γ is the following symbol and it determines its algebraic structure [10]:

$$(1)\qquad \sigma(\Gamma) : (g; \pm; [m_1, \ldots, m_r], \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k})\}),$$

where $g, k \geq 0$, $m_i, n_{i,j} \geq 2$ and every number is an integer. The quotient $\mathcal{D}/\Gamma$ has topological genus g and k boundary components. The brackets $(n_{i,1}, \ldots, n_{i,s_i})$ are called **cycle-periods** and the numbers m_i and $n_{i,j}$ are called **proper periods** and **link periods**, respectively. If $r = 0$, $k = 0$ or $s_i = 0$, we write in each respective case $[-]$, $\{-\}$, $(-)$. Also, we write m_i^t, $n_{i,j}^t$ or $(-)^t$ when a period or a cycle-period is repeated t times.

The **algebraic genus** of Γ is $p = \eta g + k - 1$, where $\eta = 2$ or 1 according to whether the sign in σ is '+' or '-'. The area of Γ is the area of any one fundamental region of Γ. It is denoted by $|\Gamma|$ and it satisfies

$$|\Gamma| = 2\pi\Big(\eta g + k - 2 + \sum\nolimits_{i=1}^{r}(1 - 1/m_i) + \frac{1}{2}\sum\nolimits_{i=1}^{k}\sum\nolimits_{j=1}^{s_i}(1 - 1/n_{i,j})\Big).$$

An NEC group Γ with signature as (1) exists if and only if $|\Gamma| > 0$ [14].

Let Γ be an NEC group with signature as (1). Γ is generated by $\{x_i\}_{i=1,\dots,r}$ elliptic transformations, $\{e_i\}_{i=1,\dots,k}$ hyperbolic transformations, $\{c_{i,j}\}_{\substack{i=1,\dots,k \\ j=0,1,\dots,s_i}}$ reflections and $\{a_i, b_i\}_{i=1,\dots,g}$ hyperbolic transformations (if the sign is '+') or $\{d_i\}_{i=1,\dots,g}$ glide reflections (if the sign is '-'). The generators satisfy the following relations:

$$x_i^{m_i} = 1, \qquad i = 1, \dots, r,$$

$$c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1, \quad i = 1, \dots, k, \ j = 1, \dots, s_i,$$

$$e_i^{-1} c_{i,0} e_i c_{i,s_i} = 1, \qquad i = 1, \dots, k,$$

$$\textstyle\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g [a_i b_i] = 1 \qquad \text{if the sign is '+'},$$

or

$$\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g d_i^2 = 1, \qquad \text{if the sign is '-'},$$

where $[a_i b_i]$ denotes the commutator $a_i b_i a_i^{-1} b_i^{-1}$.

An NEC group with sign '+' in the signature and $k = 0$ (hence $g \geq 2$) is a Fuchsian group. An NEC group which is not a Fuchsian group is called a **proper** NEC group. The subgroup of all orientation preserving elements of a proper NEC group Γ is called the **canonical Fuchsian group** of Γ and denoted by Γ^+.

Let X be a Klein surface of topological genus g and k boundary components. Then by [12] there exists an NEC group Γ with signature

$$(g; \pm; [-], \{(-)^k\}), \tag{2}$$

such that $X = \mathcal{D}/\Gamma$. In that case Γ is said to be a **surface** NEC group.

A Klein surface $X = \mathcal{D}/\Gamma$ has a canonical double cover which is the Riemann surface $X^+ = \mathcal{D}/\Gamma^+$, whose topological genus is p, the algebraic genus of X.

If $k = 0$ and sign '+', X is a classical Riemann surface; and if sign '-', X is a non-orientable Riemann surface.

G is a group of automorphisms of X with order N, if and only if there exists an NEC group Λ with $\Gamma \lhd_N \Lambda$ such that $G = \Lambda/\Gamma$ [11]. The automorphism group of X, $\mathrm{Aut}(X)$, is the quotient $N_{\mathcal{G}}(\Gamma)/\Gamma$, where the group $N_{\mathcal{G}}(\Gamma)$ denotes the normalizer of Γ in the group $\mathcal{G}$ of isometries of $\mathcal{D}$.

A set of positive integers $\{n_1, n_2, \dots, n_t\}$ satisfies the **elimination property** if

$$lcm(n_1, \dots, \widehat{n}_i, \dots, n_t) = lcm(n_1, \dots, n_t),$$

for each $i = 1, \ldots, t$. Let σ be an NEC signature as (2) and N be an odd positive integer. Given another NEC signature τ

$$(g^*; \pm; [m_1, \ldots, m_r], \{(-)^{k^*}\}), \tag{3}$$

we say that (σ, τ) is an N-pair if there exist an NEC group Λ with signature τ and an epimorphism $\theta\colon \Lambda \to \mathbb{Z}_N$ whose kernel is an NEC group Γ with signature σ. We will need later the following result [6, Th. 3.1.2 and Th. 3.1.3].

THEOREM 1: *The pair (σ, τ) is an N-pair if and only if:*

(1) *For each $i = 1, \ldots, r$, m_i divides N, and* $\mathrm{sign}(\sigma) = \mathrm{sign}(\tau)$.

(2) $|\Gamma| = N|\Lambda|$.

(3) *There exist positive divisors $l_1, \ldots, l_k$ of N such that*

 (3.1) $k = \sum_{j=1}^{k^*} N/l_j$.

 (3.2) *In the case '+' the set $\{m_1, \ldots, m_r, l_1, \ldots, l_k\}$ has the elimination property.*

(4) $N = lcm(m_1, \ldots, m_r, l_1, \ldots, l_k)$ *if $g^* = 0$ (case '+') or $g^* = 1$ (case '-').*

3. Characterization of the q-trigonality

Definition 2: (i) A Riemann (Klein) surface S of topological (algebraic) genus g (p) is called q-trigonal if and only if it admits an order three automorphism ϕ such that the quotient $S/<\phi>$ has topological (algebraic) genus q.

(ii) ϕ is called a q-trigonal automorphism of S.

The q-trigonality condition is expressed, by means of NEC groups, as follows:

PROPOSITION 3: *A Klein surface $X = \mathcal{D}/\Gamma$ of algebraic genus p is q -trigonal if and only if there exists an NEC group Γ^* of algebraic genus q such that $\Gamma \lhd_3 \Gamma^*$.*

Proof: We only need consider the results about automorphism groups shown in the previous Section. By virtue of them, there exists a q-trigonal automorphism, ϕ, of X if and only if there exists an NEC group Γ^* such that $<\phi>= \Gamma^*/\Gamma$. Furthermore, since $X/<\phi>= \mathcal{D}/\Gamma^*$ then the algebraic genus of Γ^* must be q. ∎

COROLLARY 4: *If a Klein surface $X = \mathcal{D}/\Gamma$ is q-trigonal, then the canonical Riemann surface $X^+ = \mathcal{D}/\Gamma^+$ is q-trigonal.*

Proof: Let ϕ be a q-trigonal automorphism of X. Then there exists an NEC group Γ^* of algebraic genus q such that $<\phi>= \Gamma^*/\Gamma$. If we consider the quotient of the canonical Fuchsian groups $(\Gamma^*)^+/\Gamma^+ =<\phi^+>$, we have ϕ^+ is a q-trigonal automorphism of X^+. ∎

Definition 5: An NEC group Γ^*, in the above conditions, is called a q-trigonal group of X.

The following result was obtained in [8, Th. 1].

THEOREM 6: *Let S be a q-trigonal Riemann surface of genus g. Let ϕ, ψ be q-trigonal automorphisms of S. If $g > 9q+4$ then $\phi = \psi$ or $\phi = \psi^{-1}$.*

COROLLARY 7: *Let $X = \mathcal{D}/\Gamma$ be a q-trigonal Klein surface of algebraic genus p. If $p > 9q+4$ then Γ^*, the q-trigonal group of X, is unique.*

Proof: Suppose Γ_1 and Γ_2 are q-trigonal groups of X. Then $\Gamma_1^+/\Gamma^+ =<\phi>$ and $\Gamma_2^+/\Gamma^+ =<\psi>$, where ϕ and ψ are q-trigonal automorphisms of the canonical Riemann surface $X^+ = \mathcal{D}/\Gamma^+$. The topological genus of X^+ is p, so we can apply the preceding Theorem to obtain $<\phi>=<\psi>$ and so $\Gamma_1^+ = \Gamma_2^+$.

Let $h \in \Gamma$ be an orientation reversing element; then $h \in \Gamma_1$ and $h \in \Gamma_2$. So

$$\Gamma_1 = \Gamma_1^+ \cup h\Gamma_1^+ = \Gamma_2^+ \cup h\Gamma_2^+ = \Gamma_2. \qquad ∎$$

PROPOSITION 8: *Let X be a q-trigonal Klein surface of algebraic genus $p \geq 2$ and $k \geq 0$ boundary components. Then $k - 3 \leq 3q \leq p+2$, where p and q must have the same parity.*

Proof: Let $X = \mathcal{D}/\Gamma$ where Γ has signature (2). Because X is q-trigonal there exists an NEC group Γ^* with signature (3) such that $|\Gamma| = 3|\Gamma^*|$, $m_i = 3$ and $q = \eta g^* + k^* - 1$. From the relation between areas we obtain

$$3q + 2r = p + 2, \tag{4}$$

and hence

$$3q \leq p + 2.$$

Moreover, because

$$r = \frac{p + 2 - 3q}{2}, \tag{5}$$

and r must be an integral number, we deduce that p and q have the same parity.

Let k_1 be the number of boundary components such that $\theta(e_i) = 1$, and k_3 be the number of boundary components such that $\theta(e_i) = x$, where θ is the canonical epimorphism θ: $\Gamma^* \to \mathbb{Z}_3 =< x : x^3 >$, with $\ker(\theta) = \Gamma$. Each one of the former gives three components, in Γ and each one of the latter gives one component, so that

$$k^* = k_1 + k_3, \quad k = 3k_1 + k_3;$$

then

$$\begin{aligned} k \leq & 3k^* \\ = & 3q + 3 - 3\eta g^* \\ \leq & 3q + 3. \end{aligned}$$

Furthermore, $k - k^* = 2k_1$, and then k and k^* must have the same parity. ∎

From now on Γ^* will denote an NEC group with signature

$$(g^*, \pm, [3^r], \{(-)^{k^*}\}), \tag{6}$$

where $k^* = k_1 + k_3$ is defined as above, $|\Gamma| = 3|\Gamma^*|$ and $q = \eta g^* + k^* - 1$.

Lemma 9: *Γ^* is a q-trigonal group of $X = \mathcal{D}/\Gamma$ if and only if Γ and Γ^* have the same orientability and*

(i) *if Γ is orientable then $r + k_3 \neq 1$,*
(ii) *if $g^* = 0$ then $r + k_3 \geqslant 2$,*
(iii) *if Γ is non-orientable and $g^* = 1$ then $r + k_3 \geqslant 1$.*

Proof: Γ^* is a q-trigonal group of X if and only if the pair (σ, σ^*) of signatures of Γ and Γ^* is a 3-pair. By Theorem 1, Γ and Γ^* must have the same orientability. Furthermore, in the orientable case the only thing to check is that the set $\{m_1, \dots, m_r, l_1, \dots, l_{k^*}\}$ has the elimination property. But, since $m_i = 3, i = 1, \dots, r$; $l_i = 1, i = 1, \dots, k_1$, and $l_i = 3, i = k_1 + 1, \dots, k^*$, then a necessary and sufficient condition is $r + k_3 \neq 1$. Besides, in the case $g^* = 0$, the condition (4) in Theorem 1 is equivalent to $r + k_3 \geqslant 2$. In the non-orientable case the condition $3 = lcm(m_1, \dots, m_r, l_1, \dots, l_{k^*})$ is equivalent to $r + k_3 \geqslant 1$. ∎

From now on, given two integral numbers u, v we write $par(u, v) = 0$ or $par(u, v) = \dot{1}$, according to whether u and v have the same or different parity.

For each $p \geq 2$ we denote by Q_p the set of **admissible** values q, that is, the numbers q such that there exists a q-trigonal Klein surface with algebraic genus p. The set Q_p is given in the following

Theorem 10: *The set of admissible values for each algebraic genus $p \geq 2$ is*

$$Q_p = \{q_i \in \mathbb{N} \cup \{0\} \mid q_0 \leq q_i \leq q_1 \text{ and } par(p, q_i) = 0\},$$

where q_0 is given in the following table:

	q_0	q_0	q_0
$k \equiv 0 \bmod 3$	$\max\{0, \frac{k-3}{3}\}$	$\frac{1}{3}(k+3)$	$\frac{k}{3}$
$k \equiv 1 \bmod 3$	$\frac{1}{3}(k-1)$	$\frac{1}{3}(k+5)$	$\frac{1}{3}(k+2)$
$k \equiv 2 \bmod 3$	$\frac{1}{3}(k+1)$	$\frac{1}{3}(k+7)$	$\frac{1}{3}(k+4)$
	if sign +	*if sign − and g even*	*if sign − and g odd*

$$\text{and } q_1 = \begin{cases} \frac{1}{3}(p-1) & \text{if sign + and } g \equiv 2 \bmod 3,\ k = 0 \text{ or } 1, \\ \frac{1}{3}(p+2) & \text{if otherwise.} \end{cases}$$

Proof: Let Γ be a surface NEC group with signature (2), algebraic genus $p = \eta g + k - 1$ and Γ^* as in (6). The areas relation (4) can be written as

$$\eta g + k + 1 = 3\eta g^* + 3k^* - 3 + 2r;$$

then

$$k = 3k^* - B \quad \text{where } B = \eta g - 3\eta g^* + 4 - 2r. \tag{7}$$

As we saw in Proposition 8, $k \leq 3k^*$ so $B \geq 0$. Thus g^* satisfies the condition

$$g^* \leq \frac{\eta g + 4 - 2r}{3\eta}. \tag{8}$$

Let k_1 and k_3 be as in Proposition 8. Then

$$k_1 = \frac{k - k^*}{2} = k^* - \frac{B}{2}, \quad k_3 = \frac{B}{2}. \tag{9}$$

So B must be an even integer. From (7) we have B is even if and only if $\eta g - 3\eta g^*$ is even. That always occurs if $\eta = 2$ (the orientable case), but if $\eta = 1$ (the non-orientable case) a necessary condition is that g and g^* have the same parity. By Proposition 8

$$\frac{k-3}{3} \leq q \leq \frac{1}{3}(p+2),$$

and since $par(p, q) = 0$ we can write

$$q = \frac{1}{3}(k + A) + 2l, \quad 0 \leq l \leq \frac{\eta g + 1 - A}{6}, \tag{10}$$

where $A \geq -3$ is an integer which depends on k and the parity of g.

Let $q = \frac{1}{3}(k + A) + 2l$ be an admissible value; then

$$\frac{1}{3}(k + A) + 2l = \eta g^* + k^* - 1,$$

$$k = 3\eta g^* + 3k^* - 3 - 6l - A. \tag{11}$$

From (7) and (11)

$$B = A + 3 + 6l - 3\eta g^*, \tag{12}$$

and from (12) and (7) we obtain

$$r = \frac{\eta g + 1 - A - 6l}{2}. \tag{13}$$

Our next aim is to find the smallest possible value $q \in Q_p$. For this, we see that for each possible value of $A \geq -3$ the smallest value, denoted by q_A, is given for $l = 0$, and so $q_A = \frac{1}{3}(k + A)$. Now, we proceed to get q_0:

CASE 1: If Γ is orientable, $par(q, p) = 0$ if and only if $par(k, q) = 1$; then A must be odd. Besides, $B \geq 0$ implies from (12)

$$A + 3 \geq 6g^*. \tag{14}$$

(a) If $k \equiv 0 \bmod 3$, then $A \equiv 0 \bmod 3$. Since $A \geq -3$ must be also odd, we see that the smallest value of A is $A = -3$. Now, from (14) we have $g^* = 0$, $B = 0, k = 3k^*, k_1 = k^*, k_3 = 0, r = g + 2$. So, since $k_3 + r \geq 2$, by Lemma 9 we conclude $q_0 = \frac{1}{3}(k - 3)$.

(b) If $k \equiv 1 \bmod 3$ then $A \equiv 2 \bmod 3$. Since $A \geq -3$ must be also odd, we see that the smallest value of A is $A = -1$. Now, from (14), $g^* = 0$ and so $B = 2, k_3 = 1, r = g + 1$. From Lemma 9, $q_0 = \frac{1}{3}(k - 1)$.

(c) If $k \equiv 2 \bmod 3$ then $A \equiv 1 \bmod 3$. In this case the smallest value of A is $A = 1$, and then $g^* = 0, B = 4, k_3 = 2, r = g$. Again from Lemma 9 we obtain $q_0 = \frac{1}{3}(k + 1)$.

CASE 2: If Γ is non-orientable and $par(q, p) = 0$ the study splits into:

(2.1) $par(q, k) = 0 \iff g$ is odd.

(2.2) $par(q, k) = 1 \iff g$ is even.

(2.1) If g is odd, then $par(q, k) = 0$ if and only if A is even. Furthermore, the condition $B \geq 0$ is equivalent to $A + 3 \geq 3g^*$; then $A \geq 0$.

(a) If $k \equiv 0 \bmod 3$ then $A = 0$, $g^* = 1, k_3 = 0$ and $r = \frac{1}{2}(g + 1)$. Since $g \geq 1$ then $r \geq 1$, so we can apply Lemma 9 to conclude $q_0 = k/3$.

(b) If $k \equiv 1 \bmod 3$ then $A = 2$. Here $B \geq 0$ gives $g^* = 1$ and so $k_3 = 1$ and $r = \frac{1}{2}(g-1)$. Again, from Lemma 9 we have $q_0 = \frac{1}{3}(k+2)$.

(c) If $k \equiv 2 \bmod 3$ then $A = 4$. $B \geq 0$ is satisfied if and only if $g^* = 1$. In these conditions $k_3 = 2$ and $r = \frac{1}{2}(g-3) \geq -1$. Now Lemma 9 asserts $q_0 = \frac{1}{3}(k+4)$.

(2.2) If g is even, then $par(q,k) = 1$ if and only if A is odd. Besides, as g^* must be even then $g^* \geq 2$, so $B \geq 0$ if and only if $A \geq 3$. There are no more conditions to be satisfied by Γ^*, then:

(a) If $k \equiv 0 \bmod 3$, then $A = 3$ and $q_0 = \frac{1}{3}(k+3)$.

(b) If $k \equiv 1 \bmod 3$, $A = 5$ and $q_0 = \frac{1}{3}(k+5)$.

(c) If $k \equiv 2 \bmod 3$, $A = 7$ and $q_0 = \frac{1}{3}(k+7)$.

Now, we are going to study the non-admissible values of q. To do it let us consider $q = \frac{1}{3}(k+A) + 2l$. From (9), (12) and (13) we obtain

$$k_3 + r = \frac{\eta g - 3\eta g^* + 4}{2}. \tag{15}$$

If $g^* = 0$ then $k_3 + r = g + 2 \geq 2$. If $g^* = 1$ and Γ^* is non-orientable then $k_3 + r = \frac{1}{2}(g+1) \geq 1$. So, from Lemma 9, the non-admissible cases are those for which Γ^* is orientable and $k_3 + r = 1$, that is $g^* = \frac{1}{3}(g+1)$. In particular, $g \equiv 2 \bmod 3$. If $r = 1$, then $q = p/3$; for this value we have that k_3 is necessarily equal to 0 if and only if $k = 0$. If $r = 0$, then $q = \frac{1}{3}(p+2)$; in this case k_3 is necessarily equal to 1 if and only if $k = 1$. ∎

As a consequence of the above Theorem we obtain the signatures of all q-trigonal groups

Proposition 11: *Let $X = \mathcal{D}/\Gamma$ be a Klein surface of algebraic genus $p \geq 2$, k boundary components and topological genus g. For each $q \in Q_p$ the q-trigonal groups have the following signatures:*

$$(g^*, \pm, [3^{(p+2-3q)/2}], \{(-)^{q+1-\eta g^*}\})$$

for each $g^*, 0 \leq g^* \leq (3q-k+3)/3\eta$, where $\text{sign}(\Gamma) = \text{sign}(\Gamma^*)$ *and, if* $\text{sign}(\Gamma) =$ "$-$", *then $par(g, g^*) = 0$.*

Proof: The number of periods in Γ^* is obtained from (5). The number of boundary components, k^*, comes from the fact that the algebraic genus of Γ^* is $q = \eta g^* + k^* - 1$. Now, let us consider again the areas relation

$$\eta g + k + 1 = 3\eta g^* + 3k^* - 3 + 2r.$$

Since $3k^* - k = 2k_3$, we have

$$3\eta g^* + 3k^* = \eta g + 4 - 2r,$$

so

$$3\eta g^* + 2k_3 = \eta g + 4 - 2\Big(\frac{p+2-3q}{2}\Big) = 3q - k + 3.$$

But since $k_3 \geq 0$, then $g^* \leq (3q - k + 3)/3\eta$. ∎

Let us denote by $\mathcal{K}^+_{g,k}$ (resp. $\mathcal{K}^-_{g,k}$) the family of orientable (resp. non-orientable) Klein surfaces with topological genus g and $k > 0$ boundary components. We may ask for what values $(g, k, +)$ or $(g, k, -)$ there exist admissible values. The answer is obtained as a Corollary to Theorem 10.

COROLLARY 12: *The families for which there are no admissible value q are*

$$\mathcal{K}^-_{1,3k'+2};\quad \mathcal{K}^-_{2,3k'+1};\quad \mathcal{K}^-_{2,3k'+2};\quad \mathcal{K}^-_{4,3k'+2}.$$

Proof: As we have seen in the proof of Theorem 10, the restrictions in the topological type of the surfaces, to be q-trigonal, appear in the non-orientable case. Those restrictions come from the number of proper periods, $r = \frac{1}{2}(g+1-A)$, in Γ^*, that is, the number of fixed points of a q-trigonal automorphism.

If g is odd and $k \equiv 2 \bmod 3$, then $r = \frac{1}{2}(g-3)$, so $g \geq 3$.

If g is even and $k \equiv 0, 1$ or 2, then $A = 3, 5$ or 7, respectively. So $g \geq 4$ if $k \equiv 1 \bmod 3$ and $g \geq 6$ if $k \equiv 2 \bmod 3$. ∎

It follows that in every family $\mathcal{K}^+_{g,k}$, $k > 0$ there are q-trigonal surfaces. The topological types for which there exists a unique admissible q (and so $q = q_0$) are given in the following

COROLLARY 13: *There exists a unique q if and only if g and k appear in the following table:*

	Orientable case	*Non-orientable case*
$k \equiv 0 \bmod 3$	$g = 0, 2$	$g = 1, 2, 3, 4, 6$
$k \equiv 1 \bmod 3$	$g = 0, 1, 2$	$g = 1, 3, 4, 5, 6, 8$
$k \equiv 2 \bmod 3$	$g = 0, 1, 2$	$g = 3, 5, 6, 7, 8, 10$

Proof: We need to check the cases for which the cardinality of Q_p equals 1, that is $Q_p = \{q_0\}$. This is equivalent to $q_0 + 2 > q_{\max}$, where $q_{\max} \leq q_1$ is the maximal admissible value. Now, the result follows by looking at the Table of Theorem 10. ∎

If $q = 0$, as an immediate consequence we obtain the following result which appears in [9] and [2] for bordered surfaces.

COROLLARY 14: *Let X be a cyclic trigonal Klein surface. Then the algebraic genus $p \geq 2$ of X is even and either:*

(i) *X is orientable with one or three boundary components, or*

(ii) *X is non-orientable without a boundary.*

Proof: From $k - 3 \leq 3q = 0$ we obtain $k = 1, 2, 3$. Because $par(p,q) = 0$ the algebraic genus p must be even and then k must be odd; it is $k = 1, 3$. Moreover, $q = 0$ implies that X^* is orientable and so X is too. ∎

Another easy result, as a consequence of Theorem 10 and Corollary 13, is

COROLLARY 15: *The family $\mathcal{K}^+_{0,k}$ (planar surfaces) contains q-trigonal surfaces for every $k \geq 3$ where q is unique and equal to $\frac{1}{3}(k-3)$, $\frac{1}{3}(k-1)$, or $\frac{1}{3}(k+1)$ according to whether $k \equiv 0, 1$ or $2 \bmod 3$, respectively.*

In Table 1 every topological type of bordered Klein surfaces with algebraic genus $p < 10$ appears. The middle column covers the orientable case and the right one the non-orientable case. For each topological type the admissible values of q are given.

Now let us suppose that p and q are given where $q \in Q_p$. We look for the bounds for the number of boundary components of a bordered Klein surface of algebraic genus p and q-trigonal. We have the following

PROPOSITION 16: *In the above conditions the number of boundary components k is*

(i) *Orientable case:*
If q is even: $k = 1, 3, 5, \ldots, \min\{3q+3, p+1\}$.
If q is odd: $k = 2, 4, 6, \ldots, \min\{3q+3, p+1\}$.

(ii) *Non-orientable case:*
If $par(k,p) = 0$, $k \leq \min\{p, 3q\}$.
If $par(k,p) = 1$, $k \leq \min\{p-1, 3q-3\}$.

Proof: From (10) we have $\frac{1}{3}(k+A) \leq q$, where $A \geq -3$ depends on g and k.

(i) If X is orientable A attains the lower bound, so $k \leq 3q+3$. On the other hand, since $k = p + 1 - 2g$, it follows that $k \leq \min\{3q+3, p+1\}$ and $par(k,p) = par(k,q) = 1$.

(ii) Non-orientable case. If $par(k,q) = 0$, from Theorem 10, Case 2.1, the smallest value of A is 0 and so $k \leq 3q$. If $par(k,q) = 1$, from Theorem 10, Case 2.2, $A \geq 3$ and then $k \leq 3q - 3$. In both cases $k = p + 1 - g$. Since $g \geq 1$ then $k \leq p$, but if $par(k,p) = 1$ then k must be different from p. ∎

Let us consider the family of 1-trigonal surfaces. Because $par(p,q) = 0$, p must be odd. Since $3 \cdot 1 \leq p+2$, it follows that for each p odd there exist 1-trigonal Klein surfaces. From the last Proposition we have the following Corollary.

Table 1

p	g	k	q	g	k	q
2	0	3	0	1	2	–
	1	1	0	2	1	–
3	0	4	1	1	3	1
	1	2	1	2	2	–
				3	1	1
4	0	5	2	1	4	2
	1	3	0, 2	2	3	2
	2	1	0, 2	3	2	2
				4	1	2
5	0	6	1	1	5	–
	1	4	1	2	4	–
	2	2	1	3	3	1
				4	2	–
				5	1	1
6	0	7	2	1	6	2
	1	5	2	2	5	–
	2	3	0, 2	3	4	2
	3	1	0, 2	4	3	2
				5	2	2
				6	1	2
7	0	8	3	1	7	3
	1	6	1, 3	2	6	3
	2	4	1, 3	3	5	3
	3	2	1, 3	4	4	3
				5	3	1, 3
				6	2	3
				7	1	1, 3

8	0	9	2	1	8	–
	1	7	2	2	7	–
	2	5	2	3	6	2
	3	3	0, 2	4	5	–
	4	1	0, 2	5	4	2
				6	3	2
				7	2	2
				8	1	2
9	0	10	3	1	9	3
	1	8	3	2	8	–
	2	6	1, 3	3	7	3
	3	4	1, 3	4	6	3
	4	2	1, 3	5	5	3
				6	4	3
				7	3	1, 3
				8	2	3
				9	1	1, 3

COROLLARY 17: *Let X be a bordered 1-trigonal Klein surface. Then X has odd algebraic genus. Moreover, an orientable surface X has 2, 4 or 6 boundary components and a non-orientable X has 1 or 3.*

Comments: Three classes of interesting problems to be studied on q-trigonal surfaces arise.

The first one is to find the group of automorphisms of these surfaces, for each family $\mathcal{K}^{\pm}_{g,k}$ and each q previously fixed.

The second one deals with geometrical conditions on fundamental regions of surface NEC groups. To be more precise, let X_1 and X_2 be Klein surfaces with the same topological type and the same orientability character, and let us suppose X_1 is q_1-trigonal and X_2 is q_2-trigonal, $q_1 \neq q_2$. There exist surface NEC groups Γ_1 and Γ_2 such that $X_i = \mathcal{D}/\Gamma_i$. These groups have the same signature and "similar" canonical fundamental regions R_1 and R_2. What geometrical conditions must R_1 and R_2 satisfy in order to reflect the different q_i-trigonality cases? In general, the problem may be too difficult. From Table 1, we see that the first topological type with two different values for q is $(1, 3, +)$, being $q = 0$ or 2. We think that the study of this particular family of surfaces may throw light on the general problem.

The third problem is related to the previous one. For $q \neq 0$ the quotient

$X/ < \phi >$ can have different topological types. We again think that the geometrical study of the fundamental regions would allow one to distinguish such different quotients.

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References

[1] N. L. Alling and N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture Notes in Mathematics **219**, Springer-Verlag, Berlin, 1971.

[2] E. Bujalance, J. A. Bujalance, G. Gromadzki and E. Martínez, *Cyclic trigonal Klein surfaces*, Journal of Algebra **159** (1993), 436–459.

[3] E. Bujalance and J. J. Etayo, *Characterization of q-hyperelliptic compact planar Klein surfaces*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **58** (1988), 95–104.

[4] E. Bujalance, J. J. Etayo and J. M. Gamboa, *Hyperelliptic Klein surfaces*, The Quarterly Journal of Mathematics. Oxford (2) **36** (1985), 141–157.

[5] E. Bujalance, J. J. Etayo and J. M. Gamboa, *Superficies de Klein elípticas hiperelípticas*, Memorias de la Real Academia de Ciencias, Tomo **XIX**, 1985.

[6] E. Bujalance, J. J. Etayo, J. M. Gamboa and G. Gromadzki, *Automorphism Groups of Compact Bordered Klein Surfaces. A Combinatorial Approach*, Lecture Notes in Mathematics **1439**, Springer-Verlag, Berlin, 1990.

[7] B. Estrada and E. Martınez, *On q-hyperelliptic k-bordered tori*, Glasgow Mathematical Journal **43** (2001), 343–357.

[8] H. M. Farkas, *Remarks on automorphisms of compact Riemann surfaces*, in *Discontinuous Groups and Riemann Surfaces* (L. Greenberg, ed.), Annals of Mathematics Studies, no. 79, Princeton University Press, 1974, pp. 121–144.

[9] B. H. Gross and J. Harris, *Real algebraic curves*, Annales Scientifiques de l'École Normale Supérieure **14** (1981), 157–182.

[10] A. M. Macbeath, *The classification of non-euclidean crystallographic groups*, Canadian Journal of Mathematics **19** (1967), 1192–1205.

[11] C. L. May, *Large automorphism group of compact Klein surfaces with boundary*, Glasgow Mathematical Journal **18** (1977), 1–10.

[12] R. Preston, *Projective Structures and Fundamental Domains on Compact Klein Surfaces*, Ph.D. thesis, University of Texas, 1975.

[13] H. C. Wilkie, *On non-Euclidean crystallographic groups*, Mathematische Zeitschrift **91** (1966), 87–102.

[14] H. Zieschang, E. Vogt and H. D. Coldewey, *Surfaces and planar discontinuous groups*, Lecture Notes in Mathematics **835**, Springer-Verlag, Berlin, 1980.